Work Done by a variable force

Fact (Recall) —

$$\text{Work Done} = \underbrace{\text{Force}}_{\text{component of force in the direction of travel}} \times \text{Distance}$$

$$\sum \text{Work Done} = \sum \underbrace{\text{Force}}_{\text{component of force in the direction of travel}} \times \underbrace{\delta x}_{\text{distance}}$$

$$\text{Work Done} = \int_{x_1}^{x_2} \text{Force}\, dx$$

$$\text{Work Done} = \int_{\mathbf{x}_1}^{\mathbf{x}_2} \mathbf{F} \cdot \mathbf{dx}$$

Example

A car of mass 1020 kg moves from rest at A on a horizontal surface. The driving force is constant at 1800 N and resistance to motion is modelled as $\frac{x^2}{8}$N. The car moves 120 m to B.

(a) Find the work done by the driving force and the work done against resistance as the car travels from A to B.

(b) Find the speed of the car at B

(a) $WD = 1800 \cdot 120 = 216\,\text{kJ}$. *The work done against resistance is:*

$$\int_0^{120} \frac{x^2}{8} dx = \left[\frac{x^3}{24}\right]_0^{120} = 72\,\text{kJ}$$

(b)

$$\begin{aligned} \text{Final K.E.} &= \text{Initial K.E.} + \text{Work done by driving force} - \text{Work Done against resistance} \\ &= \frac{1}{2} \cdot 1020 \cdot 0^2 + 216\,000 - 72\,000 \\ \frac{1}{2} \cdot 1020 \cdot v^2 &= 144\,000 \\ v &= 16.8\,\text{ms}^{-1} \end{aligned}$$

Example

An object is moving in a horizontal straight line against a resistive force that is directly proportional to its distance from its starting point, $f(x) = kx$. If the work done against resistance as the object travels from the origin $x = 0$ m, to a point 15 m, is 337.5 kJ.

(a) find the magnitude of k

(b) state the units of k.

$$WD = \int_0^{15} kx\,dx$$

$$337500 = k\left[\frac{x^2}{2}\right]_0^{15}$$

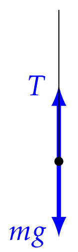
$$\Rightarrow \quad k = 3000\,\mathrm{kgs}^{-2}$$

Hooke's Law

> **Fact** (Hooke's Law) — Hooke's law states that
>
> $$\text{Force} \propto \underbrace{\text{compression}}_{\text{spring}} \;/\; \underbrace{\text{extension}}_{\text{string}}$$
>
> $$\text{Force} = \frac{\lambda \quad \times \quad x}{l}$$

What are the units of λ?

> **Example**
>
> A particle, of mass 5 kg, is suspended from a spring, of natural length 0.2 m and modulus of elasticity 40 N. Find the extension of the spring when the particle is in equilibrium



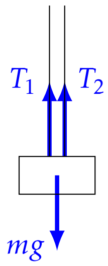
$$
\begin{aligned}
N2(\uparrow): \quad T - W &= 0 \\
\frac{\lambda x}{l} - mg &= 0 \\
\frac{40 \cdot x}{0.2} - 5g &= 0 \\
x &= \frac{1}{40} g = 0.245\,\text{m}
\end{aligned}
$$

> **Example**
>
> A light elastic spring, whcih has modulus of elasticity 85 N and natural length 1.8 m, has one end attached at a fixed point A. A horizontal force, of magnitude 40 N, is applied to the spring causing a compression. The spring rests in equilibrium, find the distance the spring is compressed from it's natural length.

$$
\begin{aligned}
N2(\rightarrow): \quad T - F &= 0 \\
\frac{85 \cdot d}{1.8} &= 40 \\
\Rightarrow \quad d &= 0.847\,\text{m}
\end{aligned}
$$

Example

An object P of mass 10 kg is attached to the lower ends of two light elastic strings. One string is of natural length 0.5 m with modulus of elasticity 25 N. The other string is of natural length 0.4 m with modulus of elasticity 30 N. The free ends of the strings are attached to a point A and P hands vertically below A. Find the distance AP.



$$
\begin{aligned}
N2(\uparrow): \quad & T_1 + T_2 - W = 0 \\
& \frac{\lambda_1(d-l_1)}{l_1} + \frac{\lambda_2(d-l_2)}{l_2} - mg = 0 \\
& \frac{25(d-0.5)}{0.5} + \frac{30(d-0.4)}{0.4} = 10g \\
& 5(d-0.5) + \frac{15}{2}(d-0.4) = g \\
& 12.5d - 5.5 = 9.8 \\
& d = 1.22\,\text{m}
\end{aligned}
$$

Example

A right circular cone C of height 4 m and base radius 3 m has its base fixed to a horizontal plane. One end of a light elastic string of natural length 2 m and modulus of elasticity 32 N is fixed to the vertex of C. The other end of the string is attached to a particle P of mass 2.5 kg. P moves in a horizontal circle with constant speed and in contact with the smooth curved surface of C. The extension of the string is 1.5 m.

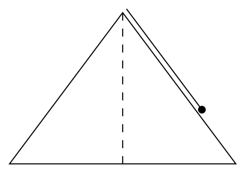
(a) Find the tension in the string. [2]

(b) Find the speed of P. [7]

(a) $T = \frac{\lambda x}{l} = \frac{32 \cdot 1.5}{2} = 24\,\text{N}$

(b)

$$
\begin{aligned}
N2(\uparrow): \quad & T\cos\theta + R\sin\theta - mg = 0 \\
\Rightarrow \quad & 24\frac{4}{5} + R\frac{3}{5} = 2.5g \\
\Rightarrow \quad & R = \frac{53}{6} \\
N2(\leftarrow): \quad & T\sin\theta - R\cos\theta = \frac{mv^2}{r} \\
& 24\frac{3}{5} - \frac{53}{6}\frac{4}{5} = \frac{2.5v^2}{3.5 \times \frac{3}{5}} \\
\Rightarrow \quad & v^2 = 6.16 \\
\Rightarrow \quad & v = 2.48\,\text{ms}^{-1}
\end{aligned}
$$



Conservation of Energy

Example

Find the work done when a light elastic string of natural length 1.2 m and modulus of elasticity 80 N is stretched from a length of 1.5 m to 1.8 m.

$$
\begin{aligned}
WD &= \int Force\, dx \\
&= \int_{1.5}^{1.8} \frac{\lambda(x-l)}{l}\, dx \\
&= \int_{0.3}^{.6} \frac{\lambda x}{1.2}\, dx \\
&= \frac{80}{1.2}\left[\frac{x^2}{2}\right]_{0.3}^{0.6} \\
&= 9\,\text{J}
\end{aligned}
$$

Fact — For an elastic object, being stretched (compressed) from x_1 to x_2 the change in elastic potential energy is $\frac{\lambda}{2l}\left(x_2^2 - x_1^2\right)$. Measuring from the natural length, we can say the EPE is $\frac{\lambda x^2}{2l}$

Example

An elastic string has natural length a. One end is fixed. A particle of mass $2m$ is attached to the free end hangs in equilibrium, with the length of the string $3a$. Find the elastic potential energy stored in the string.

$$
\begin{aligned}
N2(\uparrow): \quad & \frac{\lambda x}{l} - mg = 0 \\
\Rightarrow \quad & \lambda = \frac{mgl}{x} = \frac{2mga}{2a} = mg
\end{aligned}
$$

The energy stored is $\frac{\lambda x^2}{2l} = \frac{mg(2a)^2}{2a} = 2mga$

Example

One end of a light elastic string of natural length 0.8 m and modulus of elasticity 50 N is attached to a fixed point O. A particle P of mass 1.5 kg is attached to the other side of the string. P is released rom rest at O and falls vertically. Assuming there is no air resistance find:

(a) the extension of the string when P is at its lowest position

(b) the acceleration of P at it's lowest position

(a)

$$\text{Initial Energy} = \underbrace{0}_{\text{k.e. starts from rest}} + \underbrace{0}_{\text{e.p.e. starts unstretched}} + \underbrace{0}_{\text{g.p.e assume initial level is 0}}$$

$$\text{Final Energy} = \underbrace{0}_{\text{we are stationary when we change direction}} + \underbrace{\frac{1}{2}\frac{\lambda}{l}x^2}_{\text{e.p.e.}} + \underbrace{-mg(0.8+x)}_{\text{g.p.e assume initial level is 0}}$$

$$0 = \frac{50}{1.6}x^2 - 1.5 \cdot 9.8(x+0.8)$$

$$\Rightarrow x = 0.89219\ldots, -0.4217\ldots$$

Therefore $x = 0.89\,\text{m}$

(b) At this extension the tension is $T = \frac{50 \cdot 0.89\ldots}{0.8}$ *and weight is* $1.5g$ *so net upward force is* $\frac{50 \cdot 0.89\ldots}{0.8} - 1.5g = 41.056 = 1.5a \Rightarrow a = 27.4\,\text{ms}^{-2}$

Example

A light elastic spring, of natural length 1 m and modulus of elasticity 20 N, has one end attached to a fixed point A. A particle of mass 2 kg is attached to the other end of the spring and is held at a point B which is 0.8 m vertically below A. The particle is projected vertically downwards from B with speed $2\,\text{m}\,\text{s}^{-1}$.

Find the distance it falls before first coming to rest.

$$\text{Initial Energy} = \underbrace{\frac{1}{2}\cdot 2 \cdot 2^2}_{KE} + \underbrace{\frac{20\cdot(-0.2)^2}{2\cdot 1}}_{EPE} + \underbrace{-2g\cdot 0.8}_{GPE}$$

$$= -11.28$$

$$\text{Final Energy} = \underbrace{0}_{KE} + \underbrace{\frac{20\cdot x^2}{2\cdot 1}}_{EPE} + \underbrace{-2g\cdot(1+x)}_{GPE}$$

$$= 10x^2 - 19.6x - 19.6$$

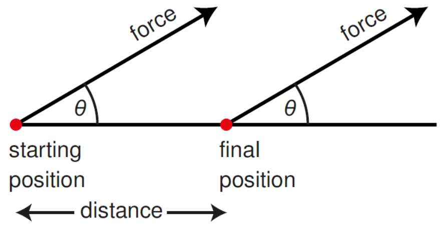
$$\Rightarrow \quad -11.28 = 10x^2 - 19.6x - 19.6$$

$$\Rightarrow \quad x = 2.32\,\text{m}$$

ie it falls 2.52 m.

Work Done, Energy and Power in two dimensions

If a constant force is directed at an angle θ to the direction of motion then:



$$\text{work done} = \text{force} \times \cos\theta \times \text{distance}$$

Fact — The work done by a (constant) force $\mathbf{F}$ which causes a displacement $\mathbf{x}$ is:

$$\text{work done} = \mathbf{F}\cdot\mathbf{x} = (|\mathbf{F}|\cos\theta)|\mathbf{x}|$$

Example

A force $(2\mathbf{i}+\mathbf{j})$N is acting on an object that moves from A, with position vector $(3\mathbf{i}+\mathbf{j})$m, to B with position vector $(4\mathbf{i}+2\mathbf{j})$m. Find the work done by the force.

$$\mathbf{x} = \begin{pmatrix}4\\2\end{pmatrix} - \begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$$

$$\mathbf{F} = \begin{pmatrix}2\\1\end{pmatrix}$$

$$\Rightarrow \quad WD = \mathbf{F}\cdot\mathbf{x}$$

$$= \begin{pmatrix}2\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\end{pmatrix} = 2+1 = 3\,\text{J}$$

Example

An object, of mass 2kg, is at rest at A when a constant force $(2\mathbf{i}-\mathbf{j})$N causes the object to move to B. A has position vector $(\mathbf{i}-2\mathbf{j})$m and B has position vector $(2\mathbf{i}-6\mathbf{j})$m. Given that no other forces act on the object, use the work-energy principle to find the speed of the object at B.

$$WD = \text{change in k.e.}$$

$$\Rightarrow \quad \mathbf{F}\cdot\mathbf{x} = \frac{1}{2}\cdot 2\cdot v^2$$

$$\Rightarrow \quad v = \sqrt{\begin{pmatrix}2\\-1\end{pmatrix}\cdot\begin{pmatrix}2-1\\-6-(-2)\end{pmatrix}}$$

$$= \sqrt{2+4} = \sqrt{6} = 2.45\,\text{ms}^{-1}$$

Definition.

- Work done on an object by a *propulsive* force is __________ in sign.
- Work done on an object by a *resistive* force is __________ in sign.

Example

A force $(-3\mathbf{i}+2\mathbf{j})$N is acting on an object that moves from A, with position vector $(\mathbf{i}-\mathbf{j})$m, to B with position vector $(5\mathbf{i}+4\mathbf{j})$m. Find the work done by the force and state whether the force is propulsive or resistive.

Fact — The kinetic energy of an object of mass m kg moving with velocity $\mathbf{v}$ ms^{-1} is defined as

kinetic energy =

where $\mathbf{v}\cdot\mathbf{v}$ is the scalar product of velocity with itself.

Example

A rocket of mass 2 tonnes is moving with velocity $(2\mathbf{i}+5\mathbf{j})$ms^{-1}. Find the kinetic energy of the rocket, giving your answer in kJ.

Example

A small object of mass 500g accelerates across a horizontal surface due to the action of a force that is acting at an angle to the resulting displacement. The driving force is $(5\mathbf{i}+3\mathbf{j})$N, and the displacement is $(65\mathbf{i}+80\mathbf{j})$m. Given that the starting speed is 4.5ms^{-1}, find the final speed of the object.

Fact — The power of an engine producing a driving force **F** N on an object moving with velocity **v** ms^{-1} can be calculated from the formula:

$$\text{power} =$$

Example

A vehicle is moving under the action of a driving force $(35\mathbf{i} - 60\mathbf{j})$ N in a horizontal plane with velocity $(5\mathbf{i} - 7\mathbf{j})ms^{-1}$. Given that no other forces are acting on the vehicle, find the power of the vehicle engine.